Delhi Public School Guwahati

"Under the aegis of the Delhi Public School Society, Delhi"

ACADEMIC UPDATE

MATHEMATICS

CLASS-XII

CH: RELATIONS AND FUNCTIONS

SYNOPSIS

- **Relation**: A relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product A × B or in other words, if A and B are two non empty sets, then a relation R from set A to set B is the subset of A×B i.e. $R: A \to B \Leftrightarrow R \subseteq A \times B$.
- **Empty relation**: A relation R in a set A is called empty relation, if no element of A is related to any element of A.
- **Universal relation**: A relation R in a set A is called universal relation, if each element of A is related to every element of A.
- A relation R in a set A is called

  Reflexive if $(a,a) \in R$, for every $a \in A$.

  Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$, for $a, b \in A$.

  Transitive if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ for $a, b, c \in A$.

- **Equivalence relation**: A relation R in a set A is called Equivalence relation if R is reflexive, symmetric and transitive.
- A function $f: A \to B$ is **one-one** (or **injective**) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall x_1, x_2 \in A$.
- A function $f: A \to B$ is **onto** (or **surjective**) if given any $y \in B$, $\exists x \in A$ such that $f(x) = y$.
- The **composition of functions** $f: A \to B$ and $g: B \to C$ is the function $gof: A \to C$

  given by $gof(x) = g(f(x) \ \forall x \in A$.

- **Some points related to composition of Functions:**
  - If f : A→ B and g : B → C are one-one, then $g \, o \, f$: A→ C is also one-one.

- If f : A→ B and g : B → C are onto, then $g\ o\ f$: A→ C is also onto.
- Let f : A→ B and g : B→ C be the given functions such that $g\ o\ f$ is one-one. Then f is one-one.
- Let f : A→ B and g : B→ C be the given functions such that $g\ o\ f$ is onto. Then g is onto.

- A function $f : A \to B$ is **invertible** if $\exists\ g : B \to A$ such that $gof = I_x$ and $fog = I_y$.
- A function $f : A \to B$ is **invertible** if f is one-one and onto.
- Let f : X → Y and g : Y → Z be two invertible functions. Then $g\ o\ f$ is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$.

Exercise

1. Let N be the set of all natural numbers and R be the relation on N×N defined by (a, b) R (c, d) iff ad (b+ c) = bc (a + d). Examine whether R is an equivalence relation on N×N.
2. Let A = {1, 2, 3}. Then show that the number of equivalence relations on A containing (2, 3) and (3, 2) is 2.
3. Show that the function $f : R \to R$ defined by $f(x) = \cos(5x + 2)$ is neither one-one nor onto.
4. Show that the relation R defined by $(a, b)R(c, d) \Leftrightarrow a + d = b + c$ for all $(a, b),(c, d) \in \mathbb{N} \times \mathbb{N}$ is an equivalence relation.
5. Show that the function $f: R \to R$ defined by $f(x) = \frac{x}{x^2+1}, \forall\ x \in R$ is neither one-one nor onto.
6. Let $\mathbb{N}$ be the set of natural numbers and let R be a relation on $\mathbb{N} \times \mathbb{N}$, defined by $(a, b)R(c, d) \Leftrightarrow ad = bc$ for all (a, b), $(c, d) \in \mathbb{N} \times \mathbb{N}$. Show that R is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.
7. Prove that function $f: \mathbb{N} \to \mathbb{N}$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto.
8. If f : R → R is defined by $f(x) = x^2 - 3x + 2$, write $f(f(x))$. (Ans: $x^4 - 6x^3 + 10x^2 - 3x$)
9. If the mappings f and g are given by f = {(1, 2), (3, 5), (4, 1)} and g = {(2, 3), (5, 1), (1,3)}, write $f\ o\ g$. (Ans: {(2, 5), (5, 2), (1, 5)})
10. Let the function f : R→ R be defined by $f(x) = \cos x$, $\forall\ x \in$ R. Show that f is neither one-one nor onto.
11. Show that the function $f: Z \to Z$ defined by $f(x) = x^2 + x$ for all $x \in Z$, is a many-one function.

12. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f: A \rightarrow B$ is mapping defined by $f(x) = \frac{x-1}{x-2}$, show that f is bijective.
13. Show that the function $f: R \rightarrow R$ given by $f(x) = ax + b$, where $a, b \in R, a \neq 0$ is a bijection.
14. If $f(x) = e^x$ and $g(x) = \log_e x$ $(x > 0)$, find *fog* and *gof*. Is *fog* = *gof* ? (Ans: *fog(x)* = *x*, *gof(x)* = *x*)
15. If the function $f: R \rightarrow R$ be given by $f(x) = x^2 + 2$ and $g: R - \{1\} \rightarrow R$ be given by $g(x) = \frac{x}{x-1}$. Find *fog* and *gof*. $\left(\text{Ans: } fog(x) = \frac{3x^2-4x+2}{(x-1)^2}, gof(x) = \frac{x^2+2}{x^2+1}\right)$
16. If $f, g : R \rightarrow R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$. Then find *fog* and *gof*. Hence find *fog*(-3), *fog*(5) and *gof*(-2). $(Ans: fog(x) = \big||x| - x\big| + |x| - x, gof(x) = \big||x| + x\big| - |x| - x, fog(-3) = 12, fog(5) = 0, gof(-2) = 4)$.
17. If $f(x) = |x|$, prove that $fof = f$.
18. Let R be a relation defined on the set of natural numbers N as $R = \{(x, y): x, y \in N, 2x + y = 41\}$. Find the domain and range of R. Also, verify whether R is reflexive, symmetric and transitive. (Ans: Domain = {1, 2, 3, …, 19, 20}, Range = {39, 37, 35, …, 7, 5, 3, 1})
19. Let R be the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b): 2 \text{ divides } (a - b)\}$. Write the equivalence class [0]. (Ans: {0, 2, 4})
20. Let $f: N \rightarrow N$ be function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f: N \rightarrow S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$. $(Ans: f^{-1} = \frac{\sqrt{y+6}-1}{3}, f^{-1}(43) = 2, f^{-1}(163) = 4)$.
21. If A = {1, 2, 3, 4 }, define relations on A which have properties of being:
    a. reflexive, transitive but not symmetric.
    b. symmetric but neither reflexive nor transitive.
    c. reflexive, symmetric and transitive.